1. Formulate a boundary value problem model of heat conduction in a slender bar of length $L$ if the left end is insulated and the right end is kept at temperature $T_0$. The initial temperature in the cross section at $x$ is $f(x)$. (5%)

2. An elastic prismatic rod is vibrated in longitudinal direction by the wave equation as the following. If at time zero, the rod is stretched by a differential deformation and released from the static state,

$$c^2 u_{xx} - u_t = 0,$$

for $t > 0$, $0 < x < L$

$$u_x(0, t) = 0, \quad u_x(L, t) = 0, \quad \text{for } t \geq 0$$

$$u(x, 0) = (1 + x)x, \quad u_t(x, 0) = 0, \quad \text{for } 0 \leq x \leq L$$

(a) Give a word statement of the constant c. (3%)

(b) Find a series solution. (15%)

3. The state of stress in a plate lying in the $x$-$y$ plane is given by $\sigma_x, \sigma_y, \sigma_{xy}$. Using Hooke's law, the strain stress relation is $\varepsilon = [S]\{\sigma\}$, where $[S]$ is the compliance matrix.

(a) Calculate the inverse matrix of $[S]$. (7%)

(b) When the strain are $\varepsilon_x = 1, \varepsilon_y = 2, \varepsilon_{xy} = 3$, and the Poisson ratio is $\nu$, find the plane stresses. (5%)

$$[S] = \begin{bmatrix} \frac{1}{E} & -\nu \frac{1}{E} & 0 \\ -\nu \frac{1}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1+\nu}{E} \end{bmatrix}$$

4. The stress state at a point in a machine element with respect to a Cartesian coordinate system is $\sigma_x = 1, \sigma_y = 1, \sigma_{xy} = -2, \sigma_z = 3, \sigma_{yz} = 0, \sigma_{zx} = 3$ (MPa).

(a) Write the stress tensor in matrix form. (5%)
(b) Compute the principle stresses and corresponding directions. (6%) 

5. Find the general solution of the equation: \( y'' = \frac{y'}{x} + \frac{x}{2y} \). (8%) 

6. Analyze the quadratic form \( f(x_1, x_2) = 3x_1^2 + 3x_2^2 + 2x_1x_2 \). 
   (a) Find the standard form of \( f(x_1, x_2) \). (5%) 
   (b) If \( f(x_1, x_2) = 4 \), draft it in the principle axes. (7%) 

7. Find the standard form of the quadratic form: \(-2x_1x_2 + 2x_1^2\). (6%) 

8. Find the general solution of \( x^2y'' + xy' + 4y = f(x) \). 
   (a) if \( f(x) = 0 \); (4%) 
   (b) if \( f(x) = \sin(2\ln x) \). (8%) 

9. Solve the differential equation: (8%) 
   \( y''' + y'' = 0 \), with \( y(0) = 0, \ y'(\pi) = 0, \ y''(\frac{\pi}{2}) = -1 \). 

10. Solve the differential equation 
    \[
    y'' + y = \begin{cases} 0, & 0 \leq t \leq \pi \\ 3\cos t, & \pi \leq t \end{cases} \text{ with } y(0) = 0. (8%) \]